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# A note on the use of approximation methods in general relativity

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**Abstract.** Using a perturbational technique, the advanced Green function of a scalar wave equation in the background of a plane gravitational wave is calculated up to first order in a parameter measuring the deviation of the metric from flat space–time. The result is shown to coincide with the corresponding expansion with respect to this parameter of the known exact Green function.

## 1. Introduction

The problematic character of perturbational methods in general relativity is sometimes pointed out by referring to the fact that the characteristics of the exact equations are always null hypersurfaces with respect to the exact metric of space–time whereas, in an iterative approach, the equations to be solved involve the zeroth order (e.g. Minkowski) metric in every order of approximation (Bird and Dixon 1975).

In this paper we consider the equation of a scalar, massless field in the background of a plane gravitational wave where an exact Green function $G$ is known (Günther 1965).

We write the metric $g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski tensor. This implies that the geometrical quantities entering the expression for $G$ are analytic in $\epsilon$. We expand the required geometrical objects and $G$ up to first order in $\epsilon$ : $G \sim \overset{0}{G} + \epsilon \overset{1}{G}$.

On the other hand, the field equations are expanded and solved up to order $\epsilon$. This gives an expression for $G$ which coincides with $\overset{0}{G} + \epsilon \overset{1}{G}$ mentioned above.

## 2. The Green function in general

The scalar wave equation in curved space–time reads

$$g^{\mu\nu}\nabla_\mu \nabla_\nu \Phi = 0. \tag{2.1}$$

(In our notation $\mu$, $\nu = 0, 1, 2, 3$. The signature of $g$ is $(+---)$. $\nabla_\mu$ means the covariant derivation with respect to $x^\mu$.)

The Green function $G$ is subject to

$$g^{1/4} g'^{1/4} g^{\mu\nu} \nabla_\mu \nabla_\nu G(x, x') = \delta^{(4)}(x, x'). \tag{2.2}$$

(Here we have defined $g = |\det g_{\mu\nu}(x)|$, $g' = |\det g_{\mu'\nu'}(x')|$ and

$$\delta^{(4)}(x, x') = \delta(x^0 - x^{0'})\, \delta(x^1 - x^{1'})\, \delta(x^2 - x^{2'})\, \delta(x^3 - x^{3'}).)$$

Now and henceforth we consider only a geodesically convex region of space-time. In such a region we can define the 'word function' $\sigma(x, x')$ as half of the geodesic distance between the points $x$ and $x'$. Furthermore,

$$\Delta(x, x') = g^{-1/2} g'^{-1/2} |\det \nabla_\mu \nabla_{\nu'} \sigma|. \tag{2.3}$$

Then the symmetric Green function $\bar{G}$ of equation (2.2) is given by

$$\bar{G} = (8\pi)^{-1}(\Delta^{1/2}(x, x')\delta(\sigma) - v(x, x')\theta(\sigma)) \tag{2.4}$$

($\theta$ denotes the step function).

In (2.4) $v$ is a smooth solution of (2.1) satisfying certain boundary conditions (for details see deWitt and Brehme 1960); $v$ vanishes in flat space–time. Its non-vanishing in general curved space–times indicates the failure of Huyghens' principle in these spaces.

One can also define an advanced Green function by

$$G^+(x, x') = 2\theta[x^{0'}, x^0]\bar{G}(x, x') \tag{2.5}$$

and a retarded one by

$$G^-(x, x') = 2\theta[x^0, x^{0'}]\bar{G}(x, x') = G^+(x', x) \tag{2.6}$$

where $\theta[x^{0'}, x^0]$ is equal to 1 when $x'$ lies in the future of $x$ and vanishes otherwise.

## 3. The world function up to order $\epsilon$ (following Synge 1960)

We have

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}. \tag{3.1}$$

The points $x$ and $x'$ are joined by a unique geodesic $\Gamma$ with respect to $g_{\mu\nu}$ and by a unique straight line $C$ with respect to $\eta_{\mu\nu}$. Let $w$ be a parameter on the latter varying from 0 to 1. Then the equation for $C$ reads

$$z^\mu = (1 - w)x^\mu + wx'^\mu. \tag{3.2}$$

From the very definition of a geodesic we have

$$\int_\Gamma g_{\mu\nu} \frac{dz^\mu}{dw} \frac{dz^\nu}{dw} dw = \int_C g_{\mu\nu} \frac{dz^\mu}{dw} \frac{dz^\nu}{dw} dw + O(\epsilon^2). \tag{3.3}$$

Hence

$$\begin{aligned}
\sigma(x, x') &= \tfrac{1}{2} \int_\Gamma g_{\mu\nu} \frac{dz^\mu}{dw} \frac{dz^\nu}{dw} dw = \tfrac{1}{2} \int_C g_{\mu\nu} \frac{dz^\mu}{dw} \frac{dz^\nu}{dw} dw + O(\epsilon^2) \\
&= \tfrac{1}{2}(x' - x)^\mu (x' - x)^\nu \int_C g_{\mu\nu}\, dw + O(\epsilon^2) \\
&= \tfrac{1}{2}(x' - x)^2 + \frac{\epsilon}{2}(x' - x)^\mu (x' - x)^\nu \int_C h_{\mu\nu}\, dw + O(\epsilon^2)
\end{aligned} \tag{3.4}$$

where $(x'-x)^2=(x'-x)^\mu(x'-x)^\nu\eta_{\mu\nu}$. It is a straightforward matter to calculate $\Delta^{1/2}$ up to order $\epsilon$ from equation (3.4).

## 4. The Green function in a plane wave background

In some coordinate systems the metric of a plane gravitational wave may be written as

$$g_{\mu\nu}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & & \\ 0 & 0 & \multicolumn{2}{c}{g_{\alpha\beta}}\end{pmatrix}\qquad \alpha,\beta=2,3 \tag{4.1}$$

where $g_{\alpha\beta}$ is a negative definite matrix with

$$g_{\alpha\beta}=g_{\alpha\beta}(u)\qquad (u=x^0-x^1).$$

Using Hadamard's theory of second-order hyperbolic equations it has been shown (Günther 1965) that, for this space–time, Huyghens' principle applies to equation (2.1), that is $v=0$. Thus

$$\bar{G}=(8\pi)^{-1}\Delta^{1/2}\delta(\sigma). \tag{4.2}$$

Up to the solution of a certain ordinary differential equation, $\sigma$ and hence $\Delta$ may be written down explicitly. This is not necessary for our purposes.

We prefer to write down $\sigma$ and $\Delta$ directly in the desired approximation. We have

$$g_{\alpha\beta}=\delta_{\alpha\beta}+\epsilon h_{\alpha\beta}. \tag{4.3}$$

From the results of § 3 we deduce

$$\sigma(x,x')=\frac{(x'-x)^2}{2}+\frac{\epsilon}{2}\frac{(x'-x)_\alpha(x'-x)_\beta}{u'-u}\int_u^{u'}h_{\alpha\beta}(u'')\,du''+O(\epsilon^2) \tag{4.4}$$

$$\Delta^{1/2}=1+\epsilon\left(\frac{1}{2(u'-u)}\int_u^{u'}h(u'')\,du''-\tfrac{1}{4}(h(u')+h(u))\right)+O(\epsilon^2) \tag{4.5}$$

where $h=\eta^{\mu\nu}h_{\mu\nu}=-\delta_{\alpha\beta}h_{\alpha\beta}$.

Taking for granted that $v(x,x')=0$

$$\begin{aligned}G^+=(4\pi)^{-1}\theta(u'-u)\bigg[&\delta\left(\frac{(x'-x)^2}{2}\right)+\frac{\epsilon}{2}\delta\left(\frac{(x'-x)^2}{2}\right)\frac{1}{u'-u}\int_u^{u'}h(u'')\,du''\\ &-\frac{\epsilon}{4}\delta\left(\frac{(x'-x)^2}{2}\right)(h(u')+h(u))+\frac{\epsilon}{2}\delta'\left(\frac{(x'-x)^2}{2}\right)\\ &\times(x'-x)_\alpha(x'-x)_\beta\frac{1}{u'-u}\int_u^{u'}h_{\alpha\beta}(u'')\,du''\bigg].\end{aligned} \tag{4.6}$$

## 5. Solving equation (2.2) up to order $\epsilon$

Substituting (4.1) into (2.2) and developing up to order $\epsilon$ yields the equation

$$\left(\Box-\epsilon h_{\alpha\beta}\partial_\alpha\partial_\beta+\epsilon\frac{\partial h}{\partial u}\partial_v\right)G=\delta^{(4)}(x,x') \tag{5.1}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ and $v = x^0 + x^1$ (not to be mixed up with the above used $v(x, x')$). We make the *ansatz*

$$G^+ = \overset{0}{G}{}^+ + \epsilon \overset{1}{G}{}^+ \tag{5.2}$$

where $\overset{0}{G}{}^+$ is the advanced Green function of flat space–time:

$$\overset{0}{G}{}^+(x, x') = (4\pi)^{-1}\theta(t'-t)\,\delta\left(\frac{(x'-x)^2}{2}\right) = (4\pi)^{-1}\theta(u'-u)\,\delta\left(\frac{(x'-x)^2}{2}\right), \tag{5.3}$$

satisfying

$$\Box \overset{0}{G}{}^+ = \delta^{(4)}(x, x'). \tag{5.4}$$

Then, again dropping terms of $O(\epsilon^2)$, equation (5.1) gives

$$\Box \overset{1}{G}{}^+ = -\frac{h}{2}\delta^{(4)}(x, x') + \left(h_{\alpha\beta}\partial_\alpha\partial_\beta - \frac{\partial h}{\partial u}\partial_v\right)\overset{0}{G}{}^+. \tag{5.5}$$

Solving (5.5) by means of $\overset{0}{G}{}^+$, we arriye at

$$\overset{1}{G}{}^+ = -\tfrac{1}{2}h(u')\overset{0}{G}{}^+ - \partial_{\alpha'}\partial_{\beta'}J_{\alpha\beta} + \partial_{v'}J \tag{5.6}$$

where

$$J(x, x') = \int d^4x''\,\overset{0}{G}{}^+(x', x'')\frac{\partial h(u'')}{\partial u''}\overset{0}{G}{}^+(x'', x) \tag{5.7}$$

$$J_{\alpha\beta}(x, x') = \int d^4x''\,\overset{0}{G}{}^+(x', x'')h_{\alpha\beta}(u'')\overset{0}{G}{}^+(x'', x). \tag{5.8}$$

The evaluation of the integrals (5.7, 8) is performed in the Appendix. Using (A.8) we obtain

$$J(x, x') = (8\pi)^{-1}\frac{\theta(u'-u)}{u'-u}\theta[(x'-x)^2](h(u') - h(u)) \tag{5.9}$$

$$J_{\alpha\beta}(x, x') = (8\pi)^{-1}\frac{\theta(u'-u)}{u'-u}\theta[(x'-x)^2]\int_u^{u'} h_{\alpha\beta}(u'')\,du''. \tag{5.10}$$

Substitution of (5.9, 10) in (5.6) yields

$$\overset{1}{G}{}^+ = (8\pi)^{-1}\theta(u'-u)\left[-\tfrac{1}{2}\delta\left(\frac{(x'-x)^2}{2}\right)(h(u') + h(u)) + \delta'\left(\frac{(x'-x)^2}{2}\right)(x'-x)_\alpha \right.$$
$$\left. \times (x'-x)_\beta\frac{1}{u'-u}\int_u^{u'} h_{\alpha\beta}(u'')\,du'' + \delta\left(\frac{(x'-x)^2}{2}\right)\frac{1}{u'-u}\int_u^{u'} h(u'')\,du''\right]. \tag{5.11}$$

Obviously, $\overset{1}{G}{}^+$, as given by (5.11) and (4.6), respectively, are identical.

Finally we remark that the procedure of this section would have worked also for the retarded Green function but not for the symmetric one. This is due to the fact that—in the latter case—the integrals corresponding to $J$, $J_{\alpha\beta}$ would contain terms which are nonzero for space-like separation of $x$ and $x'$. This difficulty can be circumvented by simply writing

$$\bar{G}(x, x') = \tfrac{1}{2}(G^-(x, x') + G^+(x, x')) = \tfrac{1}{2}(G^+(x', x) + G^+(x, x')). \qquad (5.12)$$

## Appendix

We intend to calculate the following integral:

$$H(x, x') = \int d^4x''\theta(u'-u'')\,\delta\Big(\frac{(x'-x'')^2}{2}\Big)\,\theta(u''-u)\,\delta\Big(\frac{(x''-x)^2}{2}\Big) f(u''). \qquad (A.1)$$

$f$ is an arbitrary integrable function of $u = x^0 - x^1$. Furthermore we define $v = x^0 + x^1$. Introducing polar coordinates $(\rho'', \phi'')$ in the $y''$–$z''$ plane, (A.1) may be written as

$$H(x, x') = \theta(u'-u)\int_u^{u'} du'' \int_{-\infty}^{\infty} dv'' \int_0^{\infty} d(\rho''^2) \int_0^{2\pi} d\phi''\,\delta[(u'-u'')(v'-v'') - (x'-x'')_\alpha \times (x'-x'')_\alpha]\,\delta[(u''-u)(v''-v) - (x''-x)_\alpha(x''-x)_\alpha] f(k_\mu x''^\mu) \qquad (A.2)$$

where $k^\mu$ is the null vector with components

$$k^\mu = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

A direct evaluation of the integral (A.2) is practically impossible. We may, however, use the high symmetry of $H(x, x')$ to facilitate the task considerably. By a symmetry of $H$ we mean an element $L \in \mathcal{L}_+^\uparrow$ for which $H(Lx, Lx') = H(x, x')$. It is easy to see from (A.1, 2) that this is equivalent to $Lk = k$. It is known (see, e.g., Jordan 1961) that this condition defines a three-dimensional subgroup $U$ of $\mathcal{L}_+^\uparrow$. Within $U$ there is the two-dimensional abelian subgroup of the so called 'null rotations'. Using an explicit representation of the null rotations one can show that it is possible to choose the parameters such that for the transformed vectors $Lx = \bar{x}$ and $Lx' = \bar{x}'$

$$\bar{x}_\alpha = \bar{x}'_\alpha \qquad (\alpha = 2, 3). \qquad (A.3)$$

By the definition of the $L$'s we also have

$$\bar{u} = u \qquad \bar{u}' = u'. \qquad (A.4)$$

Working in $\bar{x}$ coordinates instead of $x$ coordinates highly simplifies the integrations in (A.2). In fact,

$$H(x, x') = H(\bar{x}, \bar{x}') = \theta(u'-u)\int_u^{u'} du'' \int_{-\infty}^{\infty} d\bar{v}'' \int_0^{\infty} d(\bar{\rho}''^2) \int_0^{2\pi} d\bar{\phi}'' \times \delta[(u'-u'')(\bar{v}'-\bar{v}'') - \bar{\rho}''^2]\,\delta[(u''-u)(\bar{v}''-\bar{v}) - \bar{\rho}''^2] f(u''). \qquad (A.5)$$

A straightforward calculation yields

$$H(x,x')=2\pi\frac{\theta(u'-u)}{u'-u}\int_u^{u'}du''\,\theta\left(\frac{(u'-u'')(u''-u)(\bar{v}'-\bar{v})}{u'-u}\right)f(u'')$$

$$=2\pi\frac{\theta(u'-u)}{u'-u}\theta(\bar{v}'-\bar{v})\int_u^{u'}du''\,f(u''). \tag{A.6}$$

Using the definition of $\bar{x}$ coordinates, we have

$$(\bar{u}'-\bar{u})(\bar{v}'-\bar{v})=(u'-u)(\bar{v}'-\bar{v})=(\bar{x}'-\bar{x})^2=(x'-x)^2. \tag{A.7}$$

Hence we may write for (A.6)

$$H(x,x')=2\pi\frac{\theta(u'-u)}{u'-u}\theta[(x'-x)^2]\int_u^{u'}du''\,f(u''). \tag{A.8}$$

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